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On Continuous Two-Step Hybrid Method for Singular Initial Value Problems

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This paper discusses the formulation and implementation of non-linear method for the solution of first order singular initial value problem using collocation and interpolation of rational approximate solution. The continuous method developed is of order 8 and convergent. Numerical examples show that the method is suitable for singular problems whose initial values are not zero.

Keywords: Singular problems, rational approximate solution, interpolation, collocation, convergent

1. Introduction

Differential Equations (DEs) can describe nearly all systems undergoing change. They are ubiquitous in science and engineering as well as economics, social science, biology, business, health care, etc. Systems described by DEs are so complex, or the system that they describe are so large, that a purely analytical solution to the equations is not tractable. It is in this complex simulation that numerical methods are useful Rakesh and Ahmed [12].

Conventional numerical integrators (i.e. the Rung Kutta processes and the linear multistep formulas) whose derivation is based on polynomial approximation perform very poorly in the solution of problems possessing singularities. Methods developed using rational approximate solution have been reported to be effective for problems whose solution possess singularities Fatunla [3]. Hence, a growing interest into rational function and related topics is observed in pure mathematics, numerical analysis, theoretical physics, chemistry, mechanics and electronics. A padé approximant is that rational function whose power series expansion agrees with a prescribed power series to the highest possible order Brezinski [1].

Okosun and Ademuliyi [11] developed a three-step method for the numerical solution of ordinary differential equations with singularities using the approximate solution of the form:

$$y_{n+k} = \frac{y_n}{1+\sum_{i=1}^{k} b_j x_n^j} \tag{1}$$

And $k=2$. The scheme was based on rational functions approximation technique and its development and analysis was based on power series expansions and Dalhquist stability test method. The scheme is convergent and A-stable. Numerical results show that the scheme is accurate, effective and efficient. Lee and Stanley [6] developed a class of A-stable NLMM for solving stiff DE where they use the generalized linear multistep method given as:

$$\sum_{i=0}^{k} \alpha_i e^{Ah(k-1)} y_{n+1} = h\sum_{i=0}^{k} \varphi_{ki}(Ah) g_{n+1} \tag{2}$$

A family of NLMM was formulated and found to be A-stable in the sense of Dalhquist and are particularly effective for solving DEs whose solutions are asymptotically stable. Fatunla [4] presented a paper on NLMM for IVPs. In the form:

$$f_k(x) = \frac{A}{1+\sum_{r=1}^{k} a_r x^r} \tag{3}$$

Some k-step, k-th order nonlinear multistep methods were developed for the solution of both stiff and singular IVPs. The scheme was stable and convergent. Egbako and Adeboye [2] developed a one-step, six-order method for treating stiff differential equations using Padé approximation solution:

$$R(x)=\frac{P(x)}{q(x)} \qquad (4)$$

Where $p(x)=\sum_{i=0}^{m}a_i x^i$ and $q(x)=1+\sum_{i=0}^{n}b_i x^i$. Dalquist's model test equation was used to analyze its basic properties. Numerical results and comparative analysis with some methods was also done and it shows that the method is very efficient and more accurate. Odekunle et-al. [10] developed a class of inverse Runge-Kutta scheme for the numerical integration of singular problems, using the approximate solution of the form (2). The method was found to be L-stable and performed efficiently when applied to problems involving singularities.

Motsa and Sibanda [7] presented a paper titled numerical approach for the solution of nonlinear singular boundary value problems arising in physiology. The approach is based on application of the successive linearization method (SLM). The method was found to give accurate results comparable to results in the literature found using existing numerical methods.

2. Mathematical Background

Considering the approximate solution (4) where a_i's and b_i's are constants to be determined. (4) Can be written as;

$$R(x)=\sum_{i=0}^{m}a_i x^i - R(x)\sum_{i=1}^{n}b_i x^i \qquad (5)$$

The first derivative of (5) gives;

$$R'(x)=\sum_{i=1}^{m}i a_i x^{i-1} - \sum_{i=1}^{n}(R(x)ib_i x^{i-1}+R'(x)b_i x^i) \qquad (6)$$

Evaluating (5) at $x_{n+j},\ j=1,2,\ldots,r$ and (6) at $x_{n+j},\ j=1,2,\ldots,s$ gives a system of equations of the form;

$$U=AX \qquad (7)$$

Where

$$A=\begin{bmatrix} a_0 & a_1 & \mathrm{L} & a_m & b_1 & \mathrm{L} & b_n \end{bmatrix}^T$$

$$U=\begin{bmatrix} y_n & y_{n+1} & \mathrm{L} & y_{n+r} & f_{n+1} & \mathrm{L} & f_{n+s} \end{bmatrix}^T$$

$$X=\begin{bmatrix} 1 & \mathrm{L} & x_n^m & x_n y_n & \mathrm{L} & x_n^m y_n \\ 1 & \mathrm{L} & x_{n+1}^m & x_{n+1}y_{n+1} & \mathrm{L} & x_{n+1}^m y_{n+1} \\ \mathrm{M} & \mathrm{L} & \mathrm{M} & \mathrm{M} & \mathrm{L} & \mathrm{M} \\ 1 & \mathrm{L} & x_{n+r}^m & x_{n+r}y_{n+r} & \mathrm{L} & x_{n+r}^m y_{n+r} \\ 0 & \mathrm{L} & m x_n^{m-1} & -\left(x_n y_n' + y_n\right) & \mathrm{L} & -\left(x_n^m y_n' + m x_n^{m-1} y_n\right) \\ \mathrm{M} & \mathrm{L} & \mathrm{M} & \mathrm{M} & \mathrm{L} & \mathrm{M} \\ 0 & \mathrm{L} & m x_{n+s}^{m-1} & -\left(x_{n+s}y_{n+s}' + y_{n+s}\right) & \mathrm{L} & -\left(x_{n+s}^m y_{n+s}' + m x_{n+s}^{m-1} y_{n+s}\right) \end{bmatrix}$$

We then impose the following conditions on (5)

$$R(x_{n+j}) = y_{n+j},\ j=0,1,2,\ldots,r$$
$$R'(x_{n+j}) = f_{n+j},\ j=0,1,2,\ldots,s$$

<u>Analysis of the Method</u>

We associate the operator l with the non-linear method defined by

$$l\left[y(x) : h\right]=y_{n+t}-y\left(x_{n+t}\right)=0$$

Where $y(x)$ is an arbitrary function continuously differentiable on $[a,b]$. Following [4], we can write terms in (7) as a Taylor series expansion about the point x to obtain the expansion

$$l\left[y(x) : h\right]=c_0 y(x)+c_1 h y'(x)+\ldots+c_p h^p y^p(x)+\ldots$$

Where the constant coefficients $c_p = p = 0,1,2,\ldots$ are given as

$$c_p=\frac{1}{p!}\left[\sum_{j=1}^{r}j^p\Phi_j-\frac{1}{(p-1)!}\sum_{j=1}^{r}j^{p-1}\Psi_j-\frac{1}{(p-2)!}\sum_{j=1}^{r}j^{p-2}\gamma_j\right]$$

(7) Has order p if

$$l\left[y(x) : h\right]=0\left(h^{p+1}\right), c_0=c_1=\ldots=c_p=0, c_{p+1}\neq 0$$

Therefore c_{p+1} is the error constant and $c_{p+1}h^{p+1}y^{p+1}$ is the local truncation error (LTE).

A numerical method is said to be consistent if

(i) it has order $p\geq 1$

(ii) $\lim_{h\to 0}\left(\frac{1}{h}\left(y_{n+\omega}-y_n\right)\right)=\omega y_n'$

A numerical method is said to be zero stable if $\lim_{h\to 0} y_{n+j}=y_n$

A numerical method is said to be convergence if

(i) $\lim_{h\to 0}\left(y_{n+j}-y_n\right)\to 0$

(ii) It is consistent and zero stable

A numerical method is said to be A-stable if $\lim_{z\to\infty}\left(R(z)\right)\le 1,$ this implies that the method is bounded. $R(z)$ is the stability function, $z=\lambda h.$

Specification of the Method

In this paper, we intend to develop a two-step method with variable interval stage i.e. we consider grid points at $x_{n+j},\ j=0,u,v$ and 2. We shall consider the following as our case study.

Case I: $u=\frac{1}{2}\quad v=1\quad \omega=2$

Case II: $u=1\quad v=\frac{3}{2}\quad \omega=2$

Reasons for the Choice of Points

Because we are considering two step hybrid methods for singular initial value problems, instead of implementing the two-step at once, we considered partitioning the interval 0 to 2 into several u's and v's as seen above, the partition was taken in the sense that, in case I, the first point $\frac{1}{2}$ is closer to 0 and then 1. While in case II, the first point was taken from 1 then ascending to $\frac{2}{3}$.

Development of the Method

i. Results of the Continuous Methods

$$y_{n+\omega}=-\frac{1}{2}\left(\frac{A}{B}\right) \qquad 8$$

Where A and B are found on Appendices

ii. Results of the Discrete Methods

Case I: we considered

$u=\frac{1}{2},v=1$ and $\omega=2$.

$$y_{n+2}=\frac{\begin{gathered}-12y_n^2y_{n+1}+8y_n^2y_{n+2}+4y_n^2y_{n+\frac{1}{2}}-6hy_n^2f_{n+1}\\ +8hy_n^2f_{n+2}+hy_n^2f_{n+\frac{1}{2}}+4y_ny_{n+1}y_{n+2}\\ +8y_ny_{n+1}y_{n+\frac{1}{2}}-12y_ny_{n+2}y_{n+\frac{1}{2}}-8h^3f_nf_{n+1}f_{n+2}\\ -12h^2f_nf_{n+1}y_{n+2}+16h^2y_nf_{n+1}f_{n+2}+2h^3f_nf_{n+1}f_{n+\frac{1}{2}}\\ +6h^2f_ny_{n+1}f_{n+\frac{1}{2}}-4h^2y_nf_{n+1}f_{n+\frac{1}{2}}-6h^2y_nf_{n+2}f_{n+\frac{1}{2}}\\ -24hf_{n+1}y_{n+2}y_{n+\frac{1}{2}}+8hf_{n+2}y_{n+1}y_{n+\frac{1}{2}}+16hy_{n+1}y_{n+2}f_{n+\frac{1}{2}}\\ -8hf_ny_{n+1}y_{n+2}+22hy_nf_{n+1}y_{n+2}-8hy_nf_{n+2}y_{n+1}\\ +8hf_ny_{n+1}y_{n+\frac{1}{2}}+8hy_nf_{n+1}y_{n+\frac{1}{2}}-10hy_ny_{n+1}f_{n+\frac{1}{2}}\\ -8hy_nf_{n+2}y_{n+\frac{1}{2}}-7hy_ny_{n+2}f_{n+\frac{1}{2}}+6h^3f_{n+1}f_{n+2}f_{n+\frac{1}{2}}\\ -16h^2f_{n+1}f_{n+2}y_{n+\frac{1}{2}}+10h^2f_{n+1}y_{n+2}f_{n+\frac{1}{2}}+6h^2f_{n+2}y_{n+1}f_{n+\frac{1}{2}}\end{gathered}}{\begin{gathered}-12y_ny_{n+1}+8y_ny_{n+2}+4y_ny_{n+\frac{1}{2}}+4y_{n+1}y_{n+2}+8y_{n+1}y_{n+\frac{1}{2}}\\ -12y_{n+2}y_{n+\frac{1}{2}}+18hy_nf_{n+1}-8hf_ny_{n+2}+8h^2f_{n+1}f_{n+2}\\ +8hf_ny_{n+\frac{1}{2}}-15hy_nf_{n+\frac{1}{2}}-26h^2f_{n+1}f_{n+\frac{1}{2}}+12h^2f_nf_{n+1}\\ +6h^2f_nf_{n+\frac{1}{2}}-2hf_{n+1}y_{n+2}-16hf_{n+1}y_{n+\frac{1}{2}}+6hy_{n+1}f_{n+\frac{1}{2}}\\ +9hy_{n+2}f_{n+\frac{1}{2}}-24hyf_nf_{n+1}-8hyf_{n+1}f_{n+2}+32hyf_{n+1}f_{n+\frac{1}{2}}\end{gathered}}$$

Case II: we considered $u=1,v=\frac{3}{2}$ and $\omega=2$

$$y_{n+2}=\frac{\begin{gathered}4y_n^2y_{n+1}+8y_n^2y_{n+2}-12y_n^2y_{n+\frac{3}{2}}+2hy_n^2f_{n+1}+8hy_n^2f_{n+2}\\ -9hy_n^2f_{n+\frac{3}{2}}-12y_ny_{n+1}y_{n+2}+8y_ny_{n+1}y_{n+\frac{3}{2}}+4y_ny_{n+2}y_{n+\frac{3}{2}}\\ -8h^3f_nf_{n+1}f_{n+2}-12h^2f_nf_{n+1}y_{n+2}+6h^3f_nf_{n+1}f_{n+\frac{3}{2}}\\ +16h^2f_nf_{n+1}y_{n+\frac{3}{2}}-6h^2f_ny_{n+1}f_{n+\frac{3}{2}}-4h^2y_nf_{n+1}f_{n+\frac{3}{2}}\\ +6h^2y_nf_{n+2}f_{n+\frac{3}{2}}+8hf_{n+1}y_{n+2}y_{n+\frac{3}{2}}+8hf_{n+2}y_{n+1}y_{n+\frac{3}{2}}\\ -16hy_{n+1}y_{n+2}f_{n+\frac{3}{2}}-8hf_ny_{n+1}y_{n+2}-2hy_nf_{n+1}y_{n+2}-8hy_nf_{n+2}y_{n+1}\\ +8hf_ny_{n+1}y_{n+\frac{3}{2}}-8hy_nf_{n+1}y_{n+\frac{3}{2}}+10hy_ny_{n+1}f_{n+\frac{3}{2}}-8hy_nf_{n+2}y_{n+\frac{3}{2}}\\ +15hy_ny_{n+2}f_{n+\frac{3}{2}}+2h^3f_{n+1}f_{n+2}f_{n+\frac{3}{2}}+6h^2f_{n+1}y_{n+2}f_{n+\frac{3}{2}}\\ -6h^2f_{n+2}y_{n+1}f_{n+\frac{3}{2}}\end{gathered}}{\begin{gathered}4y_ny_{n+1}+8y_ny_{n+2}-12y_ny_{n+\frac{3}{2}}-12y_{n+1}y_{n+2}+8y_{n+1}y_{n+\frac{3}{2}}\\ +4y_{n+2}y_{n+\frac{3}{2}}-6hy_nf_{n+1}-8hf_ny_{n+2}+8hf_ny_{n+\frac{3}{2}}+7hy_nf_{n+\frac{3}{2}}\\ -46h^2f_{n+1}f_{n+\frac{3}{2}}+24h^2f_{n+2}f_{n+\frac{3}{2}}+4h^2f_nf_{n+1}+18h^2f_nf_{n+\frac{3}{2}}\\ +6hf_{n+1}y_{n+2}-6hy_{n+1}f_{n+\frac{3}{2}}-hy_{n+2}f_{n+\frac{3}{2}}-24hyf_nf_{n+\frac{3}{2}}\\ +48hyf_{n+1}f_{n+\frac{3}{2}}-24hyf_{n+2}f_{n+\frac{3}{2}}\end{gathered}}$$

3. Stability Analysis

Substituting the test equation $y'=\lambda y$ into the discrete method of case I and case II. The methods are found to be of order 8 with the following truncation errors.

Table 1. Local Truncation Error

Method	Local Truncation Error (LTE)
Case I	$\frac{1}{180}h^9\left(-40(y_n''')^3-18(y_n'')^2y_n^5-15y_n'y_n^8+12y_n'y_n'''y_n^5+60y_n''y_n'''y_n^4\right)$
Case II	$\frac{1}{180}h^9\left(-40(y_n''')^3-18(y_n'')^2y_n^5-15y_n'y_n^8+12y_n'y_n'''y_n^5+60y_n''y_n'''y_n^4\right)$

4. Numerical Examples

We consider the following problems to test the efficiency of the developed method. The following notation are used in the tables below;

$y_i :-(i=1,2)$, $y_n :-$ is the exact solution, $y :-$ is the computed result at each case, $|error|_i - (i=1,2,3,4)$ is the absolute error given by $(y-y_n)$ at each case and max $|error|_i \ (i=1,2,3,4):-$ is the maximum absolute error computed.

Example 1

$$y' = -100xy^2$$

With $y(0)=\frac{1}{51}$ and $0 \le x \le 20$. The exact solution is given as

$$y(x)=\frac{1}{1+50x^2}$$

Source: Jain [5]

Table 2. Results of Example 1

x	h	$\|error\|_1$	$\|error\|_2$	[9]	[5]
10	$\frac{1}{4}$	2.4758 (-008)	2.0380 (-008)	8.9871 (-007)	4.9371 (-008)
20		7.6786 (-009)	1.0938 (-009)	5.7320 (-008)	3.1072 (-009)

Example 2

$$y' = -10xy$$

With $y(0)=1$ and the exact solution is given as

$$y(x)=e^{-5x^2}$$

Source: Musa et-al. [8]

Table 3. Results of Example 2.

h	$\text{Max}\|error\|_1$	$\text{Max}\|error\|_2$	[8]
10^{-2}	3.6364(-003)	9.7860(-002)	1.2408 (-002)
10^{-3}	1.8291(-004)	7.0196(-003)	7.3642 (-004)
10^{-4}	3.2918(-005)	8.4217(-004)	7.0552 (-005)
10^{-5}	9.8404(-007)	9.2498(-006)	7.0335 (-006)
10^{-6}	2.1172(-008)	8.1328(-007)	7.0326 (-007)

Example 3

$$y' = 1 + y^2$$

With $y(0)=1$, $x \in [0, 0.7]$ and $h = 0.001$ the exact solution is;

$$y = \tan(t + \frac{\pi}{4})$$

Source: Ronald and Thuso [13]

Table 4: Results of Example 3.

x	Error (Case I)	Error (Case II)	[13]
0.1	3.2220 (-04)	1.0425 (-04)	3.2221 (-04)
0.2	2.4712 (-04)	2.4380 (-05)	2.4712 (-04)
0.3	1.3423 (-05)	9.3415 (-05)	1.4958 (-04)
0.4	8.8402 (-05)	2.6381 (-06)	9.3454 (-05)
0.5	9.0210 (-05)	8.2450 (-07)	5.7570 (-05)
0.6	3.7821 (-06)	5.3341 (-07)	1.0152 (-05)
0.7	1.5933 (-06)	4.2174 (-08)	1.0606 (-06)
0.8	6.7700 (-07)	1.8245 (-08)	5.5600 (-07)
0.9	4.5783 (-07)	7.7317 (-08)	2.3530 (-07)
1	3.9521 (-08)	3.4973 (-09)	1.5270 (-07)

5. Conclusion

Most problems in applied sciences, particularly in the study of vibrations, electric circuits, chemical reactions involve singularities in their solution and most of the numerical methods (i.e. LMMs) fail to handle such problems. This paper has focused on the development of "on continuous two step hybrid non-linear multistep method (NLMM) for the solution of singular IVPs". The method has been developed using Padé approximation by interpolating the approximate solution at initial grid point and collocated at some selected grid points within the interval of integration after differentiating the approximate solution. The methods obtained at all the two cases was investigated with the stability analysis properties and found to be consistent, zero stable and therefore convergent. These methods were tested on some singular problems. Table 2 gives the results comparison for example 1 between the two cases developed in this study and that of [5] and [9], in solving the particular problem, h is fixed as ¼ while x is taken at two different points (10,20). The results obtained showed that our methods performed better than that of [5] and [9]. Table 3 shows the results of our method and [8]. In executing the program, the step-length h varies as seen in the above, as h decreases the error becomes smaller. Also, Table 4 is the results and comparison between our method with [13]. In this example, the value of x is taken from $0, 0.1, ...1$. The results obtained showed that as x moves towards 1, our case I perform better than our case II and that of [13].

Finally, it was seen that methods developed using Padé approximation are effective other than using LMMs for problems whose solution possess singularities. The methods developed showed better numerical results.

Conflict of interest

The authors declare no conflict of interest.

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Appendix A

$$
\left(\begin{array}{c}
8uy_n^2 y_{n+u} - 8vy_n^2 y_{n+v} - 8uy_n^2 y_{n+\omega} + 8vy_n^2 y_{n+\omega} + 4hu^2 y_n^2 f_{n+u} - 4hv^2 y_n^2 f_{n+v} \\
+8uy_n y_{n+v} y_{n+\omega} - 8vy_n y_{n+u} y_{n+\omega} - 4uvy_n^2 y_{n+u} + 4uvy_n^2 y_{n+v} - 8huy_n^2 f_{n+\omega} \\
+8hvy_n^2 f_{n+\omega} - 8uy_n y_{n+u} y_{n+v} + 8vy_n y_{n+u} y_{n+v} - 2hu^2 vy_n^2 f_{n+u} + 2huv^2 y_n^2 f_{n+v} \\
+4h^2 t^2 f_n y_{n+u} f_{n+\omega} - 4h^2 t^2 y_n f_{n+u} f_{n+\omega} - 4h^2 t^2 f_n y_{n+v} f_{n+\omega} + 4h^2 t^2 y_n f_{n+v} f_{n+\omega} \\
+8huy_n y_{n+v} f_{n+\omega} - 8hvy_n y_{n+u} f_{n+\omega} + 4uvy_n y_{n+u} y_{n+\omega} - 4uvy_n y_{n+v} y_{n+\omega} \\
+4ht^2 f_{n+u} y_{n+v} y_{n+\omega} - 4ht^2 f_{n+v} y_{n+u} y_{n+\omega} - 4hu^2 y_n f_{n+u} y_{n+v} + 4hv^2 y_n f_{n+v} y_{n+u} \\
+4ht^2 f_n y_{n+u} y_{n+\omega} - 4ht^2 y_n f_{n+u} y_{n+\omega} - 4ht^2 f_n y_{n+v} y_{n+\omega} + 4ht^2 y_n f_{n+v} y_{n+\omega} \\
+4h^2 t^2 f_{n+u} y_{n+v} f_{n+\omega} - 4h^2 t^2 f_{n+v} y_{n+u} f_{n+\omega} + 4h^3 t^2 uf_n f_{n+u} f_{n+\omega} \\
-4h^3 t^2 vf_n f_{n+v} f_{n+\omega} + 4h^2 t^2 uf_n f_{n+u} y_{n+\omega} - 4h^2 t^2 uf_n y_{n+u} f_{n+\omega} \\
-4h^2 t^2 vf_n f_{n+v} y_{n+\omega} + 4h^2 t^2 vf_n y_{n+v} f_{n+\omega} + 4h^2 u^2 vy_n f_{n+u} f_{n+\omega} \\
-4h^2 uv^2 y_n f_{n+v} f_{n+\omega} + 2h^3 t^2 u^2 f_{n+u} f_{n+v} f_{n+\omega} - 2h^3 t^2 v^2 f_{n+u} f_{n+v} f_{n+\omega} \\
+h^2 t^2 u^2 f_{n+u} f_{n+v} y_{n+\omega} - h^2 t^2 u^2 f_{n+u} y_{n+v} f_{n+\omega} - h^2 t^2 v^2 f_{n+u} f_{n+v} y_{n+\omega} \\
+h^2 t^2 v^2 f_{n+v} y_{n+u} f_{n+\omega} + 8htuf_n y_{n+v} y_{n+\omega} - 8htuy_n f_{n+v} y_{n+\omega} - 8htvf_n y_{n+u} y_{n+\omega} \\
+8htvy_n f_{n+u} y_{n+\omega} - 8huvy_n f_{n+u} y_{n+\omega} + 8huvy_n y_{n+u} f_{n+\omega} + 8huvy_n f_{n+v} y_{n+\omega} \\
-8huvy_n y_{n+v} f_{n+\omega} + 2ht^2 uf_{n+v} y_{n+u} y_{n+\omega} - 2ht^2 uy_{n+u} y_{n+v} f_{n+\omega} - 2ht^2 vf_{n+u} y_{n+v} y_{n+\omega} \\
+2ht^2 vy_{n+u} y_{n+v} f_{n+\omega} + h^2 t^2 u^2 f_n f_{n+u} y_{n+v} - h^2 t^2 u^2 y_n f_{n+u} f_{n+v} - h^2 t^2 v^2 f_n f_{n+v} y_{n+u} \\
+h^2 t^2 v^2 y_n f_{n+u} f_{n+v} + 2ht^2 uf_n y_{n+u} y_{n+v} - 2ht^2 uy_n f_{n+v} y_{n+u} - 2ht^2 vf_n y_{n+u} y_{n+v} \\
+2ht^2 vy_n f_{n+u} y_{n+v} - 2h^3 t^2 u^2 f_n f_{n+u} f_{n+\omega} + 2h^3 t^2 v^2 f_n f_{n+v} f_{n+\omega} - h^2 t^2 u^2 f_n f_{n+u} y_{n+\omega} \\
+h^2 t^2 u^2 y_n f_{n+u} f_{n+\omega} + h^2 t^2 v^2 f_n f_{n+v} y_{n+\omega} - h^2 t^2 v^2 y_n f_{n+v} f_{n+\omega} + 8h^2 tuf_n y_{n+v} f_{n+\omega} \\
-8h^2 tuy_n f_{n+v} f_{n+\omega} - 8h^2 tvf_n y_{n+u} f_{n+\omega} + 8h^2 tvy_n f_{n+u} f_{n+\omega} - 8h^2 uvy_n f_{n+u} f_{n+\omega} \\
+8h^2 uvy_n f_{n+v} f_{n+\omega} - 2ht^2 uf_n y_{n+u} y_{n+\omega} + 2ht^2 uy_n y_{n+u} f_{n+\omega} + 2ht^2 vf_n y_{n+v} y_{n+\omega} \\
-2ht^2 vy_n y_{n+v} f_{n+\omega} + 2hu^2 vy_n f_{n+u} y_{n+\omega} - 2huv^2 y_n f_{n+v} y_{n+\omega} - 4h^3 t^2 uf_{n+u} f_{n+v} f_{n+\omega} \\
+4h^3 t^2 vf_{n+u} f_{n+v} f_{n+\omega} - 4h^2 t^2 uf_{n+u} f_{n+v} y_{n+\omega} + 4h^2 t^2 uf_{n+v} y_{n+u} f_{n+\omega} \\
+4h^2 t^2 vf_{n+u} f_{n+v} y_{n+\omega} - 4h^2 t^2 vf_{n+u} y_{n+v} f_{n+\omega} - 4h^2 tu^2 f_n f_{n+u} y_{n+v} \\
+4h^2 tu^2 y_n f_{n+u} f_{n+v} + 4h^2 tv^2 f_n f_{n+v} y_{n+u} - 4h^2 tv^2 y_n f_{n+u} f_{n+v} \\
+4h^2 uv^2 y_n f_{n+u} f_{n+v} - 4h^2 u^2 vy_n f_{n+u} f_{n+v} - 8htuf_n y_{n+u} y_{n+v} \\
+8htuy_n f_{n+v} y_{n+u} + 8htvf_n y_{n+u} y_{n+v} - 8htvy_n f_{n+u} y_{n+v} + 8huvy_n f_{n+u} y_{n+v} \\
-8huvy_n f_{n+v} y_{n+u} + 4h^3 tu^2 vf_n f_{n+u} f_{n+\omega} - 4h^3 tuv^2 f_n f_{n+v} f_{n+\omega} + 2h^2 tu^2 vf_n f_{n+u} y_{n+\omega} \\
-2h^2 tu^2 vy_n f_{n+u} f_{n+\omega} - 2h^2 tuv^2 f_n f_{n+v} y_{n+\omega} + 2h^2 tuv^2 y_n f_{n+v} f_{n+\omega} \\
+h^3 t^2 uv^2 f_{n+u} f_{n+v} f_{n+\omega} - h^3 t^2 u^2 vf_{n+u} f_{n+v} f_{n+\omega} + 4htuvf_n y_{n+u} y_{n+\omega} \\
-4htuvy_n y_{n+u} f_{n+\omega} - 4htuvf_n y_{n+v} y_{n+\omega} + 4htuvy_n y_{n+v} f_{n+\omega} \\
-h^3 t^2 uv^2 f_n f_{n+u} f_{n+v} + h^3 t^2 u^2 vf_n f_{n+u} f_{n+v} + 8h^2 tuvf_n f_{n+u} y_{n+v} \\
-8h^2 tuvf_n f_{n+v} y_{n+u} - 8h^3 tuvf_n f_{n+u} f_{n+\omega} + 8h^3 tuvf_n f_{n+v} f_{n+\omega} \\
-8h^2 tuvf_n f_{n+u} y_{n+\omega} + 8h^2 tuvf_n y_{n+u} f_{n+\omega} + 8h^2 tuvf_n f_{n+v} y_{n+\omega} \\
-8h^2 tuvf_n y_{n+v} f_{n+\omega} + 2h^2 t^2 uvf_{n+u} y_{n+v} f_{n+\omega} - 2h^2 t^2 uvf_{n+v} y_{n+u} f_{n+\omega} \\
+4h^3 tuv^2 f_n f_{n+u} f_{n+v} - 4h^3 tu^2 vf_n f_{n+u} f_{n+v} - 2h^2 t^2 uvf_n f_{n+u} y_{n+v} \\
+2h^2 t^2 uvf_n f_{n+v} y_{n+u}
\end{array}\right)
$$

Appendix B

$$
\begin{aligned}
&-4uy_n y_{n+u} + 4vy_n y_{n+v} + 4uy_n y_{n+\omega} - 4vy_n y_{n+\omega} \\
&+4uy_{n+u} y_{n+v} - 4vy_{n+u} y_{n+v} - 4uy_{n+v} y_{n+\omega} + 4vy_{n+u} y_{n+\omega} \\
&+2uvy_n y_{n+u} - 2uvy_n y_{n+v} - 2ht^2 f_{n+u} y_{n+v} + 2ht^2 f_{n+v} y_{n+u} \\
&+2hu^2 f_{n+u} y_{n+v} - 2hv^2 f_{n+v} y_{n+u} + 4huy_n f_{n+\omega} \\
&-4hvy_n f_{n+\omega} - 2ht^2 f_n y_{n+u} + 2ht^2 y_n f_{n+u} + 2ht^2 f_n y_{n+v} \\
&-2ht^2 y_n f_{n+v} - 2hu^2 y_n f_{n+u} + 2hv^2 y_n f_{n+v} - 4huy_{n+v} f_{n+\omega} \\
&+4hvy_{n+u} f_{n+\omega} - 2uvy_{n+u} y_{n+\omega} + 2uvy_{n+v} y_{n+\omega} + 2h^2 tu^2 f_n f_{n+u} \\
&-2h^2 t^2 uf_n f_{n+u} - 2h^2 tv^2 f_n f_{n+v} + 2h^2 t^2 vf_n f_{n+v} \\
&+4htuf_n y_{n+u} - 4htvf_n y_{n+v} + 2h^2 t^2 uf_n f_{n+\omega} - 2h^2 t^2 vf_n f_{n+\omega} \\
&-4htuf_n y_{n+\omega} + 4htvf_n y_{n+\omega} + 4h^2 tuf_{n+v} f_{n+\omega} - 4h^2 tvf_{n+u} f_{n+\omega} \\
&+4h^2 uvf_{n+u} f_{n+\omega} - 4h^2 uvf_{n+v} f_{n+\omega} - ht^2 uf_{n+v} y_{n+\omega} + ht^2 uy_{n+v} f_{n+\omega} \\
&+ht^2 vf_{n+u} y_{n+\omega} - ht^2 vy_{n+u} f_{n+\omega} - hu^2 vf_{n+u} y_{n+\omega} \\
&+huv^2 f_{n+v} y_{n+\omega} - ht^2 uf_n y_{n+v} + ht^2 uy_n f_{n+v} + ht^2 vf_n y_{n+u} \\
&-ht^2 vy_n f_{n+u} + hu^2 vy_n f_{n+u} - huv^2 y_n f_{n+v} - 2h^2 tu^2 f_{n+u} f_{n+v} \\
&+2h^2 t^2 uf_{n+u} f_{n+v} + 2h^2 tv^2 f_{n+u} f_{n+v} - 2h^2 t^2 vf_{n+u} f_{n+v} \\
&-2h^2 uv^2 f_{n+u} f_{n+v} + 2h^2 u^2 vf_{n+u} f_{n+v} - 4htuf_{n+v} y_{n+u} \\
&+4htvf_{n+u} y_{n+v} - 4huvf_{n+u} y_{n+v} + 4huvf_{n+v} y_{n+u} - 4h^2 tuf_n f_{n+\omega} \\
&+4h^2 tvf_n f_{n+\omega} + ht^2 uf_n y_{n+\omega} - ht^2 uy_n f_{n+\omega} - ht^2 vf_n y_{n+\omega} \\
&+ht^2 vy_n f_{n+\omega} - 2h^2 t^2 uf_{n+v} f_{n+\omega} + 2h^2 t^2 vf_{n+u} f_{n+\omega} - 2h^2 u^2 vf_{n+u} f_{n+\omega} \\
&+2h^2 uv^2 f_{n+v} f_{n+\omega} + 4htuf_{n+v} y_{n+\omega} - 4htvf_{n+u} y_{n+\omega} \\
&+4huvf_{n+u} y_{n+\omega} - 4huvy_{n+u} f_{n+\omega} - 4huvf_{n+v} y_{n+\omega} \\
&+4huvy_{n+v} f_{n+\omega} - h^2 tu^2 vf_n f_{n+u} + h^2 t^2 uvf_n f_{n+u} + h^2 tuv^2 f_n f_{n+v} \\
&-h^2 t^2 uvf_n f_{n+v} - 2htuvf_n y_{n+u} + 2htuvf_n y_{n+v} + h^2 tu^2 vf_{n+u} f_{n+\omega} \\
&-h^2 t^2 uvf_{n+u} f_{n+\omega} - h^2 tuv^2 f_{n+v} f_{n+\omega} + h^2 t^2 uvf_{n+v} f_{n+\omega} \\
&+2htuvy_{n+u} f_{n+\omega} - 2htuvy_{n+v} f_{n+\omega}
\end{aligned}
$$