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Two-Stage Improved Hybrid Methods for Integrating Special Second Ordinary Differential Equations Directly

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A set of improved numerical schemes is derived in this paper for the solution of special second order ordinary differential equations (ODEs). Convergence as well as stability analysis of the schemes is presented. Numerical experiment is performed to assess the performance of the proposed schemes. Results obtained reveal a potential superiority of the schemes over several existing methods of the same category and properties in the literature.

Keywords: Improved Numerical schemes, Hybrid method, Convergence, Numerical experiment.

1. Introduction

Modeling of many real-life problems in science and technology result to differential equations, which indicates that the importance of this set of equations cannot be overemphasized. Most at times, formulation of the models/equations is not the problem but their solutions. A model can barely be interpreted without its solution. It is a known fact that only a handful of differential equations have exact solutions proffered by analytical techniques. Hence, the need for numerical techniques to obtain the solutions approximately. Provided a solution exists for a given differential equation, obtaining an approximate form of the solution by numerical means is guaranteed. Special second order ODEs, on the other hand, are an important class of of differential equations that occur in a number of areas in sciences and engineering, for instance, quantum mechanics, molecular dynamics, celestial mechanics, quantum chemistry, astrophysics, electronics and elsewhere. Due to their importance, solution techniques aimed at obtaining their solutions became topical issue in the rank of numerical analysts in the last few decades. Some of the attempts made can be found in [1-17] and the references therein. In the immediate past decade, a class of techniques was introduced by Coleman [4], where algebraic order conditions of the methods were presented. It is the same class of methods Franco [8] named *Hybrid Method* three years after and derived explicit class of the methods. Further works done on the methods can be found in [18]. The general form of the methods can be expressed as follows:

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{m} b_i H_i, \quad i = 1,..,m,$$

$$H_i = f(x_n + c_i h, Y_i), \qquad (1)$$

$$Y_i = (1 + c_i) y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i} a_{ij} H_j,$$

where $a_{i,j}, b_i, c_i,$ are coefficients of the method; $m-1$ is the stage of the method. The method has evidently outperformed several methods designed for the same purpose in the literature. Although it is not self-starting like Runge Kutta Nyström method, but it integrates $y'' = f(x, y)$ pretty independent of y'_n, and it has less number of stages. But the less stages of this class of methods does not translate to the expected efficiency sometimes. This motivates a new class of methods in this paper as follows:

$$2y_{n+1} = 3y_n - y_{n-2} + \frac{(2h)^2}{2} \sum_{i=1}^{m} b_i H_i, \quad i = 1,..,m,$$

$$H_i = f(x_n + c_i h, Y_i), \qquad (2)$$

$$2Y_i = (2+c_i)y_n - c_i y_{n-2} + 2h^2 \sum_{j=1}^{i} a_{ij} H_j.$$

The method can be summarized in Butcher-like tableau as follows:

Table 1. General Coefficients

-2	0				
0	0	0			
c_3	$a_{3,1}$	$a_{3,2}$	0		
c_4	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	0	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
c_m	$a_{m,1}$	$a_{m,2}$	$a_{m,3}$	$\ldots$	$a_{m,m}$
	b_1	b_2	b_3	$\ldots$	b_m

The general form of the problem of interest is

$$y''(x) = f(x, y), \; y(x_0) = y_0, \; y'(x_0) = y'_0, \quad (3)$$

which is an initial value problem based on special second order ODE.

The remaining part of the paper is organized as follows: in Section 2, stability analysis of the method is presented. Order conditions for convergence of the methods are presented in Section 3. Derivation of the proposed method is presented in Section 4 Numerical test results of the proposed methods are presented in Section 5. Conclusion is given in Section 6.

2. Stability Analysis

One of the most important properties of numerical schemes for differential equations is stability. In this section, absolute stability property of the proposed methods is analyzed.

Consider a scalar test equation

$$y'' = (\omega)^2 \lambda^2, \quad \lambda > 0. \quad (4)$$

To study the stability property of the methods, it suffices to apply eqn. (2) to eq. (4), which yields

$$y_{n+1} = S\left(z^2\right) y_n - P\left(z^2\right) y_{n-2}. \quad (5)$$

The characteristics equation resulting from the difference equation above is

$$r^3 - S\left(z^2\right) r^2 + P\left(z^2\right), \quad (6)$$

which is the stability polynomial of the method. Where

$$S\left(z^2\right) = \frac{3}{2} - \frac{1}{2} z^2 \left(2\mathrm{e} + \mathrm{c}\right)\left(\mathrm{I} + z^2 \mathrm{A}\right)^{-1} \mathrm{b}^T,$$

$$P\left(z^2\right) = \frac{1}{2} - \frac{1}{2} z^2 \left(\mathrm{c}\right)\left(\mathrm{I} + z^2 \mathrm{A}\right)^{-1} \mathrm{b}^T,$$

$z = h\lambda$, $\mathrm{A} = \left[a_{i,j}\right]$, $\mathrm{b} = [b_1, \ldots, b_m]^T$, $\mathrm{c} = [c_1, \ldots, c_m]^T$, $\mathrm{e} = [1, \ldots, 1]^T$ and I is an $m \times m$ identity matrix.

3. Order Conditions

Order conditions of a numerical scheme are relationships between coefficients of the scheme that cause annihilation of successive terms in the Taylor series of its local truncation error, [4]. Local truncation error on the other hand is the difference between true and approximate solutions. Let $y(x)$ be the true solution of eq.(3), the local truncation error of the method (2) can be defined as

$$LTE_{n+1} = y(x_{n+1}) - y_{n+1}, \quad (7)$$

y_{n+1} is the approximate solution produced by (2). We obtain the Taylor series of (7) as

$$LTE_{n+1} = 2h^2\left(3 - 2\sum_{i=1}^{m} b_i\right) F_1^{(2)} - 2h^3\left(1 + 2\sum_{i=1}^{m} b_i c_i\right) F_1^{(3)} +$$

$$8h^3\left(3+8\sum_{i=1}^{m}b_ic_i-8\sum_{i=1}^{m}b_ia_{i,j}\right)F_2^{(3)}+O(h^4).$$
(8)

The order conditions up to order four are obtained from (8) as

$$Order1:\quad \sum b_i=\frac{3}{2},$$

$$Order2:\quad \sum b_ic_i=-\frac{1}{2},$$

$$Order3:\quad \sum b_ic_i^2=\frac{3}{4},\sum b_ia_{ij}=-\frac{1}{8},$$

$$Order4:\quad \sum b_ic_i^3=-\frac{3}{4},\sum b_ic_ia_{ij}=\frac{3}{8},\sum b_ia_{ij}c_j=\frac{5}{24}.$$

3.1 Order of Convergence

The proposed method (2) is convergent of order, say p, if

$$LTE_{n+1}=O(h^{p+2}).\qquad (9)$$

4. Derivation of the Proposed Method

Here, the proposed method is specified. The method is a three-step method and is multistage in nature. It is a third order method with two function evaluation per step.

To achieve these, equations from third order conditions for convergence are solved, which results to

$$a_{3,2}=-a_{3,1}+c_3+\frac{1}{2}c_3{}^2,\ b_3=-\frac{1}{4}\frac{1}{c_3(2+c_3)},$$

$$b_2=\frac{1}{8}\frac{10c_3+1}{c_3},\ b_1=\frac{1}{8}\frac{3+2c_3}{2+c_3}.$$

Apparently, the coefficients depend on two parameters that can freely be chosen to completely obtain the method. Although the degree of freedom is guided by a requirement that error norm of the method is minimized. Error norm is defined by

$$E_{nrm}=\left\|\tau_i^{(p+2)}\right\|.$$

There are two choices made, resulting to two methods, namely; three-step third order hybrid method(1) ThHM3(1) and three-step third order hybrid method (2) ThHM3(2). That is,

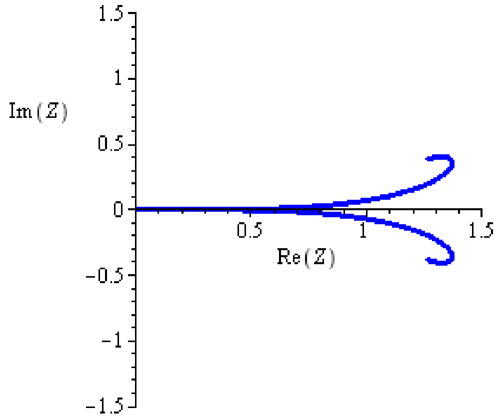
Table 2. Coefficients of ThHM3(1)

-2	0		
0	0	0	
-3	0	$\frac{3}{2}$	0
	$\frac{3}{8}$	$\frac{29}{24}$	$-\frac{1}{12}$

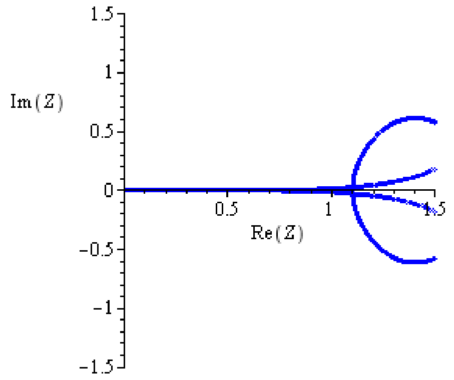
Table 3. Coefficients of ThHM3(2)

-2	0		
0	0	0	
-3	$\frac{5}{4}$	$\frac{1}{4}$	0
	$\frac{3}{8}$	$\frac{29}{24}$	$-\frac{1}{12}$

with error norms and intervals of stability 2.08×10^{-01}, 1.01×10^{-11}; $(0,1.4)$ and $(0,1.6)$, respectively.



Stability curve of ThHM3(1)



Stability curve of ThHM3(2)

5. Numerical Results

Having derived the proposed schemes, it is pertinent to test their performance. This section is devoted to that. The test is conducted by solving a set of model problems obtainable in the literature.

5.1 Test Problem

Problem 1 (Homogeneous Problem), Source: [15]

$$\frac{d^2y(x)}{dx^2} = -y(x), \quad y(0)=0, \quad y'(0)=1.$$

$$y(x)=\sin(x), \quad x\in[0,10].$$

Problem 2 (Inhomogeneous Problem), Source: [1]

$$\frac{d^2y(x)}{dx^2} = -y(x)+x, y(0)=1, y'(0)=2.$$

$$y(x)=\sin(x)+\cos(x)+x, \quad [0,10]$$

Problem 3 (Duffing Problem), Source: [18]

$$y'' = F\cos(vx)-y-y^3, \quad x\in[0,100],$$

$$F=0.002 \quad \text{and} \quad v=1.01,$$

$$y(0)=0.200426728067, \quad y'(0)=0,$$

$$y(x)=\sum_{i=0}^{4}\theta_{2i+1}\cos[(2i+1)vx], \quad [0,10]$$

$$\theta_1=0.200179477536, \theta_3=0.246946143\times10^{-3},$$

$$\theta_5=0.304014\times10^{-6}, \theta_7=0.374\times10^{-9}, \text{and} \theta_9<10^{-12}.$$

- **ThHM3(1)**: (I) Three-step third order hybrid method derived in Section 4 of the paper.
- **ThHM3(2)**: (II) Three-step third order hybrid method derived in Section 4 of the paper.
- **CRK3**: Third order classical Runge-Kutta method obtained in [6].

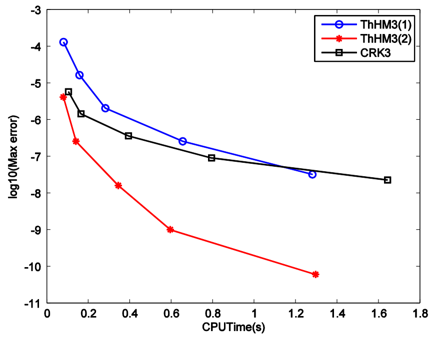


Figure 1. Efficiency curves for problem 1

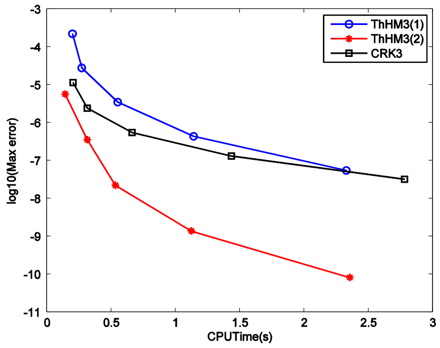


Figure 2. Efficiency curves for problem 2

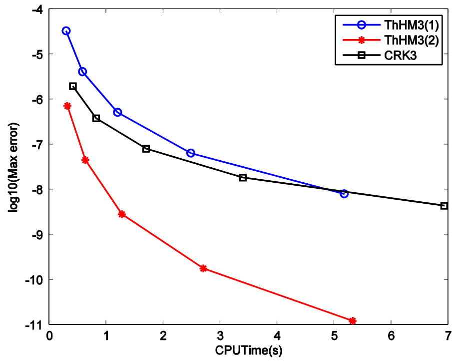


Figure 3. Efficiency curves for problem 2

Figures 1-3 show the efficiency of the proposed methods as compared with existing third order classical Runge-Kutta method. It can be observed from the figures that the proposed method, especially ThHM3(2), lies below every other curves, and it appears shorter than CRK3 curve. It shows that the ThHM3(2) integrates the problems with better accuracy and efficiency than CRK3. On the other hand, ThHM3(1) is more efficient in solving the problems than the other methods, but with less accuracy, since it appears shortest and lies above every other curve in the figures.

6. Conclusion

A class of Three-step hybrid methods is developed in this paper, where two-stage third order methods are presented. A new set of algebraic order conditions of the methods are presented. Convergence as well as stability of the methods are analyzed. Numerical experiment reveals a potential superiority of the methods over existing methods.

Conflict of interest

The authors declare no conflict of interest.

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