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Department of Mathematic, Delta State University, Abraka

***Corresponding author's email:**

ojoborsun@delsu.edu.ng

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A Perturbed Orthogonal Collocation Approach for the Solution of Fredholm Integro-Differential Equations

Sunday A. Ojobor and Efeturi M. Aleke

This paper considers the numerical solution of Fredholm integro-differential equations via a perturbed orthogonal collocation approach (POCA) with Chebyshev polynomials as trial functions. Here, the original problem is slightly perturbed by a perturbation term, $H_n(x)$, in the form of a trial solution. The perturbed problem is then evaluated and solved by collocating at the zeros of the Chebyshev polynomials to generate an algebraic linear system of equations, which on solving via the Gaussian elimination method yields the unknown parameters, which generates the required approximate solution. Here, numerical evidence obtained via POCA was compared with the orthogonal collocation method (OCM) and exact solution as available in the literature. MAPLE 18 software was used for all computational frameworks in this research.

Keywords: Chebyshev polynomials, Collocation method, Fredholm integro-differential equation, Integral equation, perturbed orthogonal collocation approach.

1. Introduction

A general Fredholm integro-differential equation of the form (Wazwaz, 2011)

$$u^{(i)}(x) = f(x) + \lambda \int_a^b k(x,s)u(s)ds, x \in [\alpha, \beta]$$

where i denotes the ith derivative, $k(x,s)$ is the kernel, and $f(x)$ is a non homogeneous function, $\lambda > 0$. This type of equation is mostly applicable in predictive control in AP monitor ([4]), the temperature distribution in the cylindrical conductor ([13]), dynamic optimization ([2]; [6]), etc.

Many of these equations are difficult to handle analytically. Hence, numerical methods have become more renowned in solving these equations. Over the years, several numerical methods for integro-differential equations have been proposed in the literature. Some of these include, the variational iteration method (see for instance, [7]; [3]), the homotopy perturbation method ([14]), the power series approximation method ([9]), the Tau and Tau-collocation methods method ([5]; [8]), the Adomian decomposition method ([11], [1]), etc.

In this paper, our concern is to apply with the development of a collocation approach with certain orthogonal basis functions for the numerical treatment Fredholm integro-differential equations. The collocation method has found extension application in recent years and a wealth of the results obtained around in the literature. For examples, [10] applied a collocation method with different collocation points for the numerical solution of second-order boundary value problem. [9] constructed a class of continuous orthogonal polynomials and were employed in an orthogonal collocation method for seeking the approximate solution of Fredholm integro-differential equations (FIDES).

Collocation at the roots of orthogonal polynomials is the called orthogonal collocation. We seek in this paper in particular, the numerical solution of the Fredholm integro-differential equation in a perturbed orthogonal collocation approach (POCA) using the Chebyshev polynomials of the first kind as basis/trial functions in the approximation of the analytic solution.

2. Chebyshev Polynomial of the First Kind

This is defined as

$$T_r(x) = \cos(r\cos^{-1}x) = x \in [-1,1] \tag{1}$$

and the recurrence relation

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x) \qquad (2)$$

satisfying the initial conditions

$$T_0(x) = 1, and\ T_1(x) = x.$$

2.1 Orthogonal Properties of Chebyshev Polynomials of First Kind

Here, we can show that

$$\int_{-1}^{1} \frac{T_r(x)T_s(x)}{(1-x^2)^{\frac{1}{2}}} dx = \begin{cases} 0, r \neq s \\ \frac{\pi}{2}, r = s \neq 0 \\ \pi, r = s = 0 \end{cases}$$

See [12] for proof.

3. Orthogonal Collocation Method and Perturbed Orthogonal Collocation Approach for the Solution of Fredholm Integro – Differential Equation

We consider the class of Chebyshev polynomials for approximating the analytic solution of integro-differential equations. This we do by approximating analytic solution $u(x)$ by

$$u_n(x) = \sum_{r=0}^{n} a_r T_r(x) \cong u(x)$$

with the orthogonal polynomials defined in section 2 above.

Now, slightly perturbing equation (1) as follows:

$$u^{(i)}(x) - \lambda \int_a^b k(x,s)u(s)\,ds, = f(x) + H_n(x) \quad (3)$$

where $a \leq x, s \leq b$, $H_n(x)$ is the perturbation term given as

$$H_n(x) = \sum_{r=0}^{n} \tau_r T_r(x) \qquad (4)$$

We write the approximate solution as:

$$u_n(x) = \sum_{r=0}^{n} a_r T_r(x) \cong u(x) \qquad (5)$$

where $T_r(x), r \geq 0$ is the rth Chebyshev polynomial and $a_r, r \geq 0$ are constants to be determined.

Substituting (4) and (5) in (3) and interchanging the order of integration and summation in the integral term, we obtain

$$\sum_{r=0}^{n} a_r T_r^{(i)}(x) - \lambda \sum_{j=0}^{n} a_r \int_a^b k(x,s) T_j(s)\,ds = f(x) + \sum_{j=0}^{n} \tau_r T_r(x), \qquad (6)$$

where $f(x)$ is a known function and occurs to the first degree, and the kernel k(x,s) is also known. We can now collocate at the zeros of $T_r(x)$ in equation (6) and obtain a set of $(n+1)$ equations in $(n+1)$ unknown comprising of $a_r, r \geq 0$ and τ. A matrix solver, which in this case, is the Gaussian elimination method is employed to solve the resulting linear algebraic equations for a unique determination of the unknown coefficients in the approximate. Substituting the known coefficients in (5) generates the required approximate solution for the problem (3).

The above methodology for the solution of equation (1) implies the perturbed orthogonal collocation approach (POCA). However, when solving equation (1) unperturbed using the same methodology as above implies orthogonal collocation method (OCM).

4. Numerical Perspectives

Here in this section, we consider the perturbed orthogonal collocation approach (POCA) for seeking the solution of linear Fredholm integro-differential equations (FIDEs) with Chebyshev basis functions. Here, a given problem is solved for various values of n (say $n = 3, 4$ and 5) using both the orthogonal collocation method (OCM) and POCA for comparison purposes.

The error formulation for this problem is defined as

$$e_r = |u(x) - u_i(x)|$$

where $u(x)$ and $u_i(x), i = 1,2,3,\ldots,n,$ are the exact and approximate solutions respectively.

Example 1 [11] Consider the linear Fredholm integro-differential equations of the form

$$u'(x) = 3 + 6x + x\int_0^1 tu(t)\,dt, \qquad u(0) = 1 \qquad (7)$$

The analytic solution is given as

$$u(x) = 3x + 4x^2 \quad (8)$$

Let the approximate solution be given as

$$u(x) = \sum_{i=1}^{n} a_i T_i(x), \quad x \in [-1,1] \quad (9)$$

By OCM, we have for $n = 3$,

$$u(x) = a_0 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) \quad (10)$$

Substituting (10) into (7) to get

$$a_1 + 4a_2 x + a_2(12x^2 - 3) = 3 + 6x + x\int_0^1 t\left(a_0 + a_1 t + a_2(2t^2 - 1) + a_3(4t^3 - 3t)\right)dt, \quad (11)$$

Chebyshev polynomials are orthogonal in the interval [-1,1].

$$t = \frac{s+1}{2}, \quad dt = \frac{1}{2}ds \quad (12)$$

Substituting (12) into (11), we have

$$a_1 + 4a_2 x + a_2(12x^2 - 3) = 3 + 6x + x\int_{-1}^{1}\left(\frac{s+1}{4}\right)\left(a_0 + a_1\left(\frac{s+1}{2}\right) + a_2\left(2\left(\frac{s+1}{2}\right)^2 - 1\right) + a_3\left(4\left(\frac{s+1}{2}\right)^3 - 3\left(\frac{s+1}{2}\right)\right)\right)ds, \quad (13)$$

Evaluating (4.9) using MAPLE 18, we get

$$a_1 + 4a_2 x + a_3(12x^2 - 3) = 3 + 6x + x\left(-\frac{1}{5}a_3 + \frac{1}{3}a_1 + \frac{1}{2}a_0\right) \quad (14)$$

Since there are four (4) constants in (14), we collocate at the zero of $T_4(x)$, that is,

$x = 0.3826834325, -0.3826834325, 0.9238795325, -0.9238795325$

to get the linear systems of equations given in the form:

$$Ax = b, \quad (15)$$

where,

$$A = \begin{pmatrix} 0.8724388558 & 1.530733730 & -1.166104000 & -0.1913417162 \\ 1.127561144 & -1..530733730 & -1.319177372 & 0.1913417162 \\ 0.6920401558 & 3.695518130 & 7.427416592 & -0.4619397662 \\ 1.307959844 & -3.695518130 & 7.057864780 & 0.4619397662 \end{pmatrix}$$

,

$x = (a_0, a_1, a_2, a_3)$,

$b = (-5.296100595, 0.703899405, 8.543277195, -2.543277195)$.

Thus, solving (15) by Gaussian elimination method (GEM), we get the following estimates:

$a_0 = 2.000000$, $a_1 = 3.000000$, $a_2 = 2.000000$, $a_3 = 0.000000$

Substituting the above estimates, $a_i, i = 0(1)3$, into (9), we get

$$u(x) = 3x + 4x^2,$$

which is the analytic solution.

Also, solving the same problem (7) using the POCA as follows:

We slightly perturbed (7) with Chebyshev polynomials of degree 3, that is,

$$\left(\sum_{i=1}^{3} a_i T_i(x)\right)' = 3 + 6x + x\int_0^1 t\left(\sum_{i=1}^{3} a_i T_i(t)\right)dt + H_n(x)$$

where

$$H_n(x) = \sum_{i=1}^{3} \tau_i T_i(x) \quad (n = 3)$$

$$\Rightarrow \left(\sum_{i=1}^{3} a_i T_i(x)\right)' = 3 + 6x + x\int_0^1 t\left(\sum_{i=1}^{3} a_i T_i(t)\right)dt + \sum_{i=1}^{3} \tau_i T_i(x) \quad (16)$$

$$\Rightarrow \left(\sum_{i=1}^{3} a_i T_i(x)\right)' = 3 + 6x + x\int_{-1}^{1}\left(\frac{s+1}{4}\right)\left(\sum_{i=1}^{3} a_i T_i\left(\frac{s+1}{2}\right)\right)ds + \sum_{i=1}^{3} \tau_i T_i(x) \quad (17)$$

Evaluating (17) using MAPLE 18, we get

$$a_1 + 4a_2 x + a_3(12x^2 - 3) = 3 + 6x + \frac{1}{2}x\left(-\frac{2}{5}a_3 + \frac{2}{3}a_1 + a_0\right) + \tau_1 x + \tau_2(2x^2 - 1) + \tau_3(4x^3 - 3x) \quad (18)$$

Since there are seven constants, we collocate at the zeros of $T_7(x)$, that is,
$x = 0$, 0.9749279125, -0.9749279125, 0.4338837390, -0.4338837390, 0.7818314824, -0.7818314824, as follows:

Collocating at $x = 0.000000$, we have:

$$a_1 - 3.a_3 + \tau_2 - 3 = 0$$

Collocating at $x = 0.974928$, we have:

$$-0.4874639562a_0 + 0.6750240292a_1 + 3.899711650a_2 + 8.600798802a_3$$
$$-0.9749279125\tau_1 - 0.900968869\tau_2 - 0.7818314849\tau_3 - 8.849567475 = 0$$

Collocating at $x = -0.974928$, we have:

$$0.4874639562a_0 + 1.324975971a_1 - 3.899711650a_2 + 8.210827638a_3$$
$$+0.9749279125\tau_1 - 0.900968869\tau_2 + 0.7818314849\tau_3 + 2.849567475 = 0$$

Collocating at $x = 0.433884$, we have:

$$-0.2169418695a_0 + 0.8553720870a_1 + 1.735534956a_2 - 0.6541620642a_3$$
$$-0.4338837390\tau_1 + 0.6234898020\tau_2 + 0.9749279120\tau_3 \text{ [illegible] } = 0$$

Collocating at $x = -0.433884$, we have:

$$0.2169418695a_0 + 1.144627913a_1 - 1.735534956a_2 - 0.8277155598a_3$$
$$0.4338837390\tau_1 + 0.6234898020\tau_2 - 0.9749279120\tau_3 - 0.3986697366 = 0$$

Collocating at $x = 0.781831$ we have:

$$-0.3909157412a_0 + 0.7393895058a_1 + 3.127325930a_2 + 4.491491900a_3$$
$$-0.7818314824\tau_1 - 0.222520934\tau_2 + 0.4338837390\tau_3 - 7.690988894 = 0$$

Collocating at $x = -0.781831$, we have:

$$0.3909157412a_0 + 1.260610494a_1 - 3.127325930a_2 + 4.178759306a_3$$
$$+0.7818314824\tau_1 - 0.222520934\tau_2 - 0.4338837390\tau_3 + 1.690988894 = 0$$

Solving the above system of equations by Gaussian elimination method (GEM) via MAPLE 18 software, we get the following estimates

$a_0 = 2.000000$, $a_1 = 3.000000$, $a_2 = 2.000000$, $a_3 = 0.000000$, $\tau_1 = \tau_2 = \tau_3 = 0.000000$.

Substituting the above estimates, $a_i, i = 0(1)3$, into (9), we get

$$u(x) = 3x + 4x^2,$$

which is the analytic solution.

Example 2 [11] Consider the linear Fredholm integro-differential equations of the form

$$u''(x) = \frac{5}{3} - 11x + \int_{-1}^{1}(xt^2 - x^2t)u(t)\,dt, \quad u(0) = 1, \; u'(0) = 1 \qquad (19)$$

The analytic solution is given as

$$u(x) = 1 + x + \frac{5}{6}x^2 \equiv \frac{5}{9}x^2. \qquad (20)$$

Using same methodology above, we have the following results:

For OCM at $n = 5$:

The approximate solution is

$$u(x) = 1.416667 + 0.250001x + 0.416667(2x^2 - 1) - 0.416667(4x^3 - 3x)$$

.

See computational results in Table 1 below.

Table 1. Comparison of results between the exact and OCM

x	Exact	OCM	Error
0.10	1.1066667	1.1066667	0.0000e+00
0.20	1.2200000	1.2200000	0.0000e+00
0.30	1.3300000	1.3300000	0.0000e+00
0.40	1.4266667	1.4266667	0.0000e+00
0.50	1.5000000	1.5000000	1.0000e-09
0.60	1.5400000	1.5400000	0.0000e+00
0.70	1.5366667	1.5366667	0.0000e+00
0.80	1.4800000	1.4800000	1.0000e-09
0.90	1.3600000	1.3600000	0.0000e+00
1.00	1.1666667	1.1666667	0.0000e+00

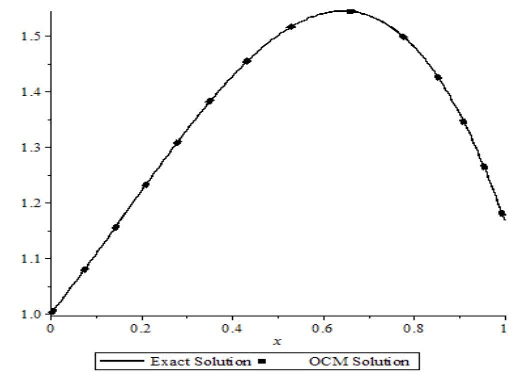

Figure 1a. Graphical simulation for Example 2 using OCM

For POCA at $n = 5$:

The approximate solution is

$$u(x) = 1.416667 + 2.250001x + 0.416667(2x^2 - 1) - 0.416667(4x^3 - 3x)$$

.

See computational results in Table 2 below.

Table 2. Comparison of results between the exact and POCA

x	Exact	POCA	Error
0.10	1.1066667	1.1066667	0.0000e+00
0.20	1.2200000	1.2200000	0.0000e+00
0.30	1.3300000	1.3300000	0.0000e+00
0.40	1.4266667	1.4266667	0.0000e+00
0.50	1.5000000	1.5000000	1.0000e-09
0.60	1.5400000	1.5400000	0.0000e+00
0.70	1.5366667	1.5366667	0.0000e+00
0.80	1.4800000	1.4800000	1.0000e-09
0.90	1.3600000	1.3600000	0.0000e+00
1.00	1.1666667	1.1666667	0.0000e+00

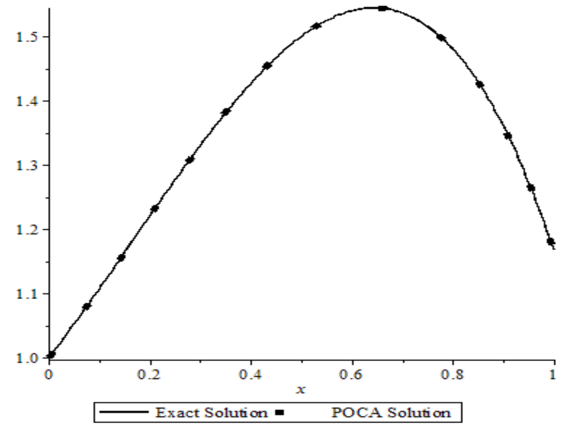

Figure 1b. Graphical simulation for Example 2 using POCA

5. Discussion of results

We have implemented the perturbed orthogonal collocation approach (POCA) for the numerical solutions of linear Ferdholm integro-differential equations of various order. Chebyshev polynomials were employed as trial functions in the approximation of the analytic solution. The orthogonal collocation method (OCM) and perturbed orthogonal collocation approach (POCA) were both used to solve the given problems for comparison purposes. In Example 1, the OCM produced an approximation that converges rapidly to the analytic solution as found in the literature at the interpolation point n = 3. In like manner, the POCA when applied to the same problem produces the analytic solution at the same interpolation point n = 3. However, the OCM and POCA produced the analytic solution at different collocation points. In Example 2, the OCM produced the required approximation at n = 5. However, the maximum error incurred is of order 10^{-9}. On using the POCA, the minimum error of order 10^{-9} was also obtained.

6. Conclusion

The observations captioned in executing both the OCM and POCA clearly depict that both converges equally. This implies that whether the problem is perturbed or unperturbed, same result is produced. This is most notable in the graphical simulation of Example 2, as shown in figure 1a and 1b respectively.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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